# Representing Smooth Boundary and Isotropic Tetrahedral Mesh Generation

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## Overview

**We propose an algorithm to generate an isotropic tetrahedral mesh whose domain boundaries are smooth.Input is a surface mesh whose inside is the meshing domain.**

**First we generate an isotropic mesh by optimal Delaunay triangulation. Then select boundary vertices and move them to new positions for better expression of the domain boundaries.**

**We get new positions using quadric error metric (QEM). To prevent the isotropy from getting worse, we determine movement distance of each vertex based on the isotropy of incident elements.**

## Investigators


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## Optimal Delaunay triangulation [Chen 04]

Use optimal Delaunay triangulation to decide the positions of inside vertices.

$$\min \quad E_{\mathrm{ODT}} = \frac{1}{n+1}\sum_{i=1}^{N}\int_{\Omega_i}\|x-v_i\|^2 dx$$

$n$ : dimension

$N$ : number of vertices

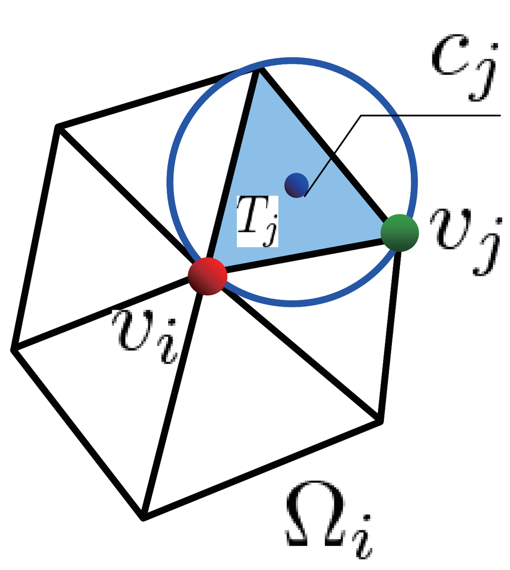
$\Omega_i$ : star of $v_i$

Vertex $v_i$ 's optimal position is

$$v_i^* = \frac{\sum_{v_j \in N(v_i)} |T_j| c_j}{\sum_{v_j \in N(v_i)} |T_j|}.$$

$|T_j|$ : volume of tetrahedron $T_j$

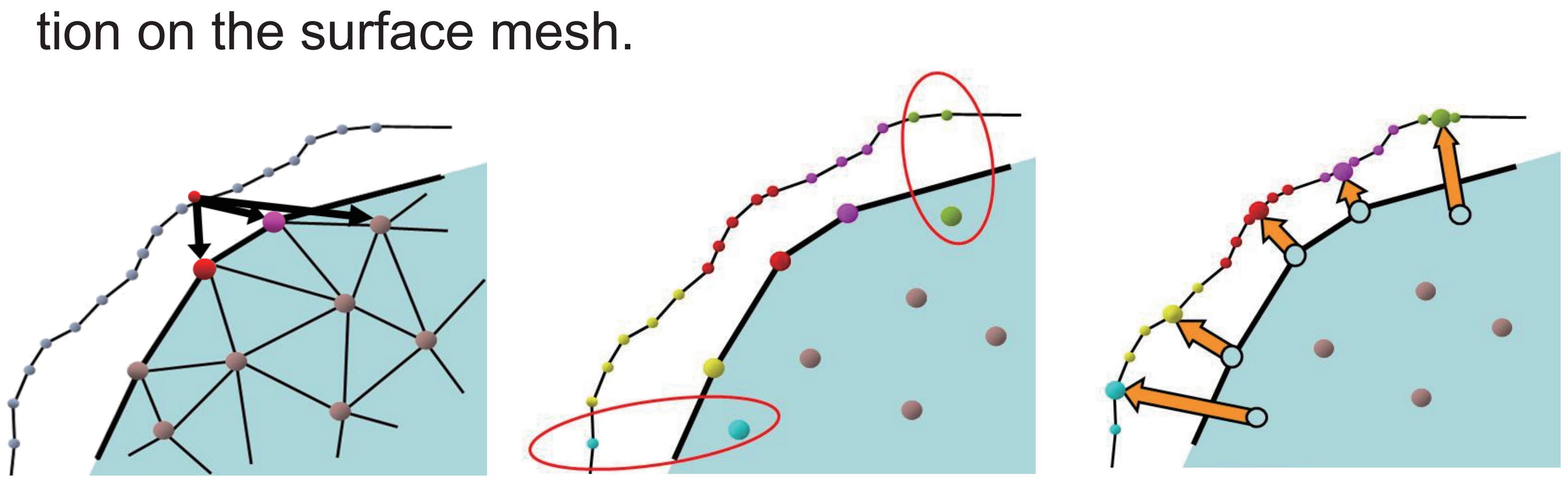
$c_j$ : circumcenter of tetrahedron $T_j$




## Selection of boundary vertices

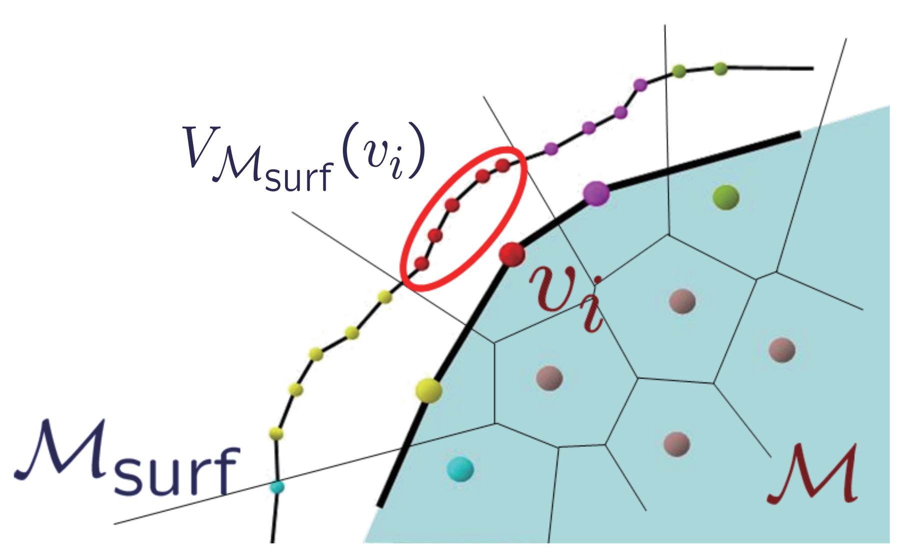
Select a vertex of tetrahedral mesh that is the nearest one from each vertex of surface mesh.

Move vertices selected in the previous step to the nearest position on the surface mesh.



It works as a discrete Voronoi diagram.

$$\boxed{V_{\mathcal{M}_{\mathrm{surf}}}(v_i) \approx \text{(Voronoi region of } v_i) \cap \mathcal{M}_{\mathrm{surf}}}$$




## Quadric error metric (QEM)

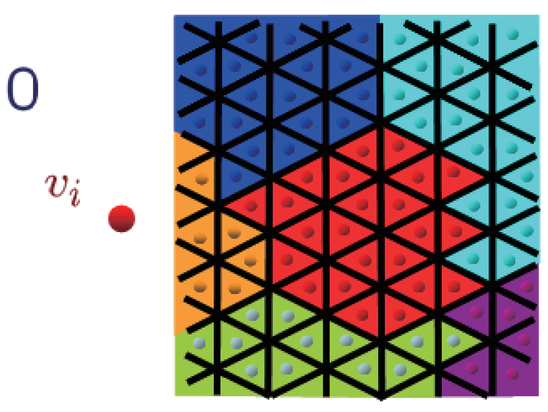
Error of a vertex $v_i = (x_i, y_i, z_i)$ to $F(v_i)$, a set of faces of surface mesh, is as follows.

$$\boxed{\Delta(v_i) = \sum_{f \in F(v_i)} (f^{\mathrm{T}} \tilde{v}_i)^2}$$

$f: \; ax + by + cz + d = 0$

$f = (a, b, c, d)^{\mathrm{T}}$

$\tilde{v}_i = (x_i, y_i, z_i, 1)^{\mathrm{T}}$



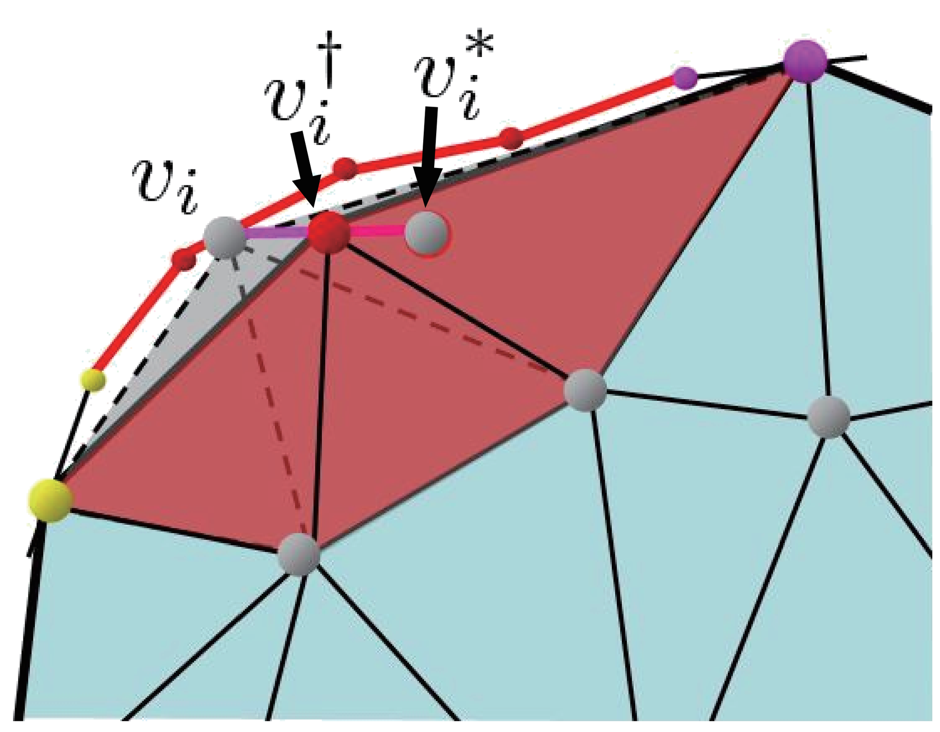

To minimize the volume of difference between the surface mesh and the tetrahedral mesh, weight above metric with the areas of faces of the surface mesh.

$$\Delta'(v_i) = \tilde{v}_i^{\mathrm{T}} \left( \sum_{f \in F(v_i)} |f| \begin{pmatrix} a^2 & ab & ac & ad \\ ba & b^2 & bc & bd \\ ca & cb & c^2 & cd \\ da & db & dc & d^2 \end{pmatrix} \right) \tilde{v}_i$$

## Movement of boundary vertices

Move the boundary vertices to the new positions.

Not to worsen the tetrahedral isotropy, new positions are decided by binary search.



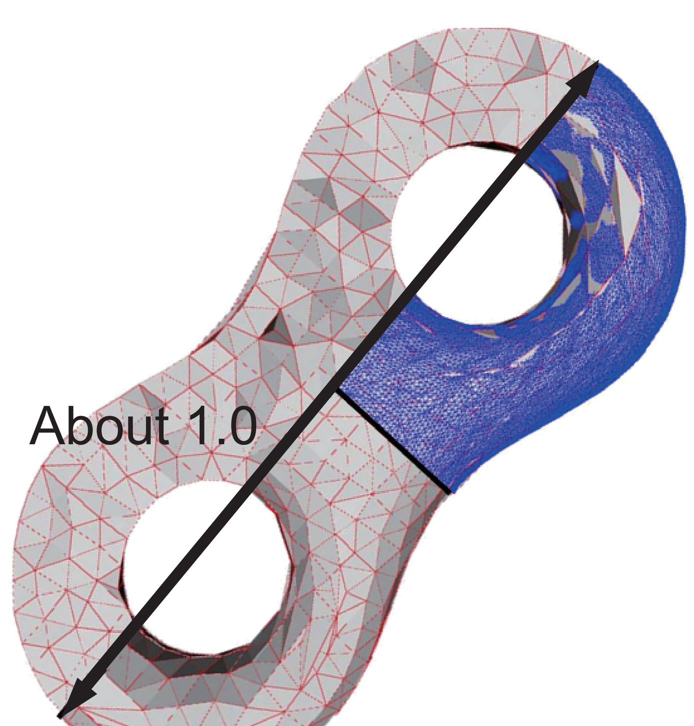

$$\boxed{\alpha Q(v_i) \leq Q(v_i^\dagger)}$$

$Q(v)$ : isotropy of tetrahedra incident to a vertex $v$ , $0 \leq Q(v) \leq 1$

$v_i^\dagger$ : candidate destination of $v_i$

$\alpha$ : user-specified parameter, $0 \leq \alpha \leq 1$

## Result




| | Isotropy | Average of QEM ($\times 10^{-8}$ ) |
|---|---|---|
| Our method | 0.751 | 2.68 |
| Centroid + projection | 0.742 | 2.77 |
| No-postprocessing | 0.742 | 2.96 |

Comparison (centroid + projection): move vertices to the centroids of $V_{\mathcal{M}_{\mathrm{surf}}}(v_i)$. Then move them to the nearest positions on the surface mesh (project).